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How to get masses from Kaluza-Klein theory

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Abstract. *In the general Kaluza-Klein theory on a principal bundle* P *with structure group* G *, the vertical part of the metric* g *on* P *is defined by a scalar field* γ *on* P *. We consider* γ *as a Higgs field but, instead of looking for an appropriate potential, we constrain this scalar field to a* G *-orbit. This procedure provides symmetry breaking and the Yang-Mills part of the fields split into a residual Yang-Mills field and a set of vector bosons which are shown to be massive. We write down the field equations for gravity, Yang-Mills and massive gauge bosons. An example with internal symmetry group* Spin(4) *broken to* $U(1)$ *is worked out.*

1. INTRODUCTION

The theory proposed by T. Kaluza [1] and O. Klein [2] for the unification of gravitation and electromagnetism admits a simple and elegant geometrical interpretation: the electromagnetic potential corresponds to a connection on a principal bundle M_5 with structure group $U(1)$ and base M_4 (space-time) [3] and it can be combined with the gravitation field to form a lorentzian metric on M_5 . The field equations for the unified theory are then derived from Einstein equations for the metric on M_5 .

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The Kaluza-Klein theory was first generalized by R. Kerner [4] (see also Y. Cho [5]) to a principal bundles with compact non-abelian structure groups in order to have a unified theory of gravitation and other physical interactions (which are described by means of Yang-Mills fields). It has been remarked by Y. Cho and G. Freund [6] that scalar bosons can be naturally incorporated to this theory and might give symmetry breaking provided they could be considered as Higgs fields. Y. Cho and G. Freund failed to find an appropriate «Higgs potential» and their idea was considered later by J. Scherk and J. Schwarz [7] who could only obtain symmetry breaking in a special case (when the structure group is a so-called «flat group») which do not seem to appear naturally in physics (for other approaches see [8], [9], [10]).

The last generalization of this picture has been done, through the introduction of certain lorentzian metrics on associated bundles, by E. Witten [11] whose paper led to an increase of interest in the Kaluza-Klein theory (a computer search gave 285 papers published from 1982 to 1987).

In this paper we analyse the symmetry breaking in the Kaluza-Klein theory on a principal fibre bundle by considering the vertical part of the metric as a Higgs field, the values of which are constrained to an orbit of the symmetry group. This approach is related to the interpretation of the Higgs mechanism given in [12] where it has been shown that constraining a scalar field to an orbit is the basic underlying procedure for the creation of massive vector bosons. Howevere the situation we will describe below does not satisfy all the usual assumptions for the Higgs mechanism [13] [14] [15]: classically, an invariant scalar product is chosen on the Lie algebra of the (compact) symmetry group and a Higgs field is a scalar field with values in a hermitian space in which is given a unitary representation of the symmetry group. In the present paper we are working with a varying non-invariant scalar product on the Lie algebra which is precisely considered as the Higgs field (associated to the coadjoint representation).

As in the classical case, we obtain a splitting of the gauge field into a residual Yang-Mills field and vector bosons which are shown to be a massive. This splitting is related to a foliation of the principal bundle by principal subbundles naturally defined by the Higgs field. Using this foliation we compute the mass matrix of the vector bosons in terms of the kinetic energy of the Higgs field.

The field equations (generalized Einstein equations) decompose into ordinary Einstein equation, Yang-Mills equation and Klein-Gordon type equation for the scalar field. The constraint condition on the Higgs field ensures that the last equation is a consequence of the second one.

The general theory is illustrated by an example where the symmetry group is $\mathrm{Spin}(4)$ and is reduced to $U(1)$. An imbedding of $SU(2) \times U(1)$ into $\mathrm{Spin}(4)$ could enable us to interprete the results in terms of the standard model. We obtain two pairs of charged massive vector bosons, a neutral massive vector boson and electromagnetic field as well as gravitation.

The mechanism of symmetry breaking described here does not allow for a Yukawa coupling to fermions and it does not give rise to fermion masses. It could be used to described the first stage of symmetry breaking in grand unified theories in which fermions remain massless. The second stage can be described in terms of an analogous mechanism discussed in a subsequent paper [16].

2. INVARIANT METRIC ON A PRINCIPAL FIBRE BUNDLE

2.1. Canonical splitting

Let $\pi : P \to M$ be a principal fibre bundle with structure group G and consider a G-invariant lorentzian metric g on P with signature $(-,+,+,\dots,+)$, such that the restriction of g to the vertical tangent space at any point of P is positive-definite. We denote by g the Lie algebra of G.

Let

$$\gamma : P \to \otimes_s^2 g^*$$

be the map given by:

$$\gamma(p)(\xi,\eta) = g(\xi_P,\eta_P)(p) \tag{2.1}$$

where $p \in P$, $\xi,\eta \in g$ and ξ_P,η_P are the fundamental vector fields on P associated to ξ,η. It is easy to see that:

(i) $\gamma(p)$ is a euclidean (positive-definite) scalar product on g;

(ii) the map γ is equivariant, i.e.:

$$\gamma(pa) = Ad^*(a)\gamma(p) \tag{2.2}$$

for each $p \in P$ and each $a \in G$.

Let ω be the1-form on P with values in g defined by:

$$\gamma(p)(\omega(p)(X),\xi) = g(p)(X,\xi_P(p)) \tag{2.3}$$

for each $p \in P$, $X \in T_pP$ and $\xi \in g$.

From (2.1), (2.3) and the G-invariance of g, one deduces that ω is a connection form on the principal bundle P [17], [18].

Consider the covariant, symmetric tensor field h on P defined by:

$$h(p)(X,Y) = \gamma(p)(\omega(p)(X),\omega(p)(Y))$$

$$p \in P, \quad X,Y \in T_pP.$$

h is obviously G-invariant and $h(p)(X,Y) = 0$ if one of the vectors X and Y is vertical.

Hence there exists a unique metric tensor $\hat{g}$ on M such that:

$$h = \pi^*\hat{g}.$$

It is clear that $\hat{g}$ is a lorentzian metric. Then we can write

$$g(p)(X,Y) = \pi^*\hat{g}(p)(X,Y) + \gamma(p)(\omega(p)(X),\omega(p)(Y)). \tag{2.4}$$

This formula will be called the *canonical splitting* of g .

2.2. Ricci tensor

In this section we compute the Ricci tensor and the scalar curvature of the invariant metric g given before, using the canonoical splitting (2.4).

It will be convenient to introduce a local frame field

$$(X_\alpha, X_i)_{\substack{1\leq\alpha\leq n \\ n+1\leq i\leq n+N}} \qquad \text{where } n = \dim M,\ N = \dim G,$$

defined on an open set $\pi^{-1}(U)$ in P , where U is a coordinate neighbourhood in M with coordinates $(x^\alpha)_{1\leq\alpha\leq n}$. The vector fields $X_\alpha(\alpha = 1,\dots n)$ are the horizontal lifts, with respect to the connection ω , of the natural vector fields $\frac{\partial}{\partial x^\alpha}$ on U .

Let $(e_{n+1},\dots,e_{n+N})$ be a basis of the Lie algebra g of G. Then the vector fields $X_i = (e_i)_P$ $(i = n+1,\dots,n+N)$ are the fundamental vertical vector fields on P associated with this basis. We have:

$$g_{\alpha\beta} = g(X_\alpha, X_\beta) = \hat{g}\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)\circ\pi = \hat{g}_{\alpha\beta}\circ\pi \tag{2.5}$$

$$g(X_\alpha, X_i) = 0 \tag{2.6}$$

$$g(X_i, X_j) = \gamma_{ij} \qquad \text{with } \gamma_{ij}(p) = \gamma(p)(e_i, e_j), p\in P. \tag{2.7}$$

The frame field (X_α, X_i) is non-holonomic and the brackets can be expressed in terms of the curvature form Ω of the connection ω and of the structure constants of the Lie algebra g as follows:

$$[X_\alpha, X_\beta] = -\Omega^\ell{}_{\alpha\beta}X_\ell \tag{2.8}$$

$$[X_\alpha, X_i] = 0. \tag{2.8}$$

$$[X_i, X_j] = c^k{}_{ij}X_k \tag{2.10}$$

with $\Omega(X_\alpha, X_\beta) = \Omega^\ell{}_{\alpha\beta}e_\ell$, $[e_i, e_j] = c^\ell{}_{ij}e_\ell$.

Let ∇ denotes the covariant differential on P with respect to the Levi-Civita connection of g. We have:

2.11
$$\begin{aligned}2g(\nabla_X Y, Z) =& X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) - \\ & -g((X,[Y,Z]) - g([X,Z],Y) + g([X,Y],Z).\end{aligned}$$

Substituting the formulae (2.5) through (2.10) into (2.11) one gets:

(2.12)
$$\nabla_{X_\alpha} X_\beta = \hat{\Gamma}^{\delta}{}_{\beta\alpha} \circ \pi \, X_\delta - \frac{1}{2}\Omega^{\ell}{}_{\alpha\beta} X_\ell$$

(2.13)
$$\nabla_{X_\alpha} X_i = \nabla_{X_i} X_\alpha = \frac{1}{2}\gamma_{i\ell}\Omega^{\ell}{}_{\alpha\gamma} g^{\lambda\delta} X_\delta + \frac{1}{2}\gamma^{\ell k} D_\alpha \gamma_{ik} X_\ell$$

(2.14)
$$\nabla_{X_i} X_j = -\frac{1}{2} g^{\delta\mu} D_\mu \gamma_{ij} X_\delta + \Gamma^{\ell}{}_{ij} X_\ell$$

where:

(i) $\hat{\Gamma}^{\delta}{}_{\alpha\beta}$ are Christoffel symbols for the metric $\hat{g}$, i.e.

$$\hat{\nabla}_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = \hat{\Gamma}_{\beta\alpha} \frac{\partial}{\partial x^\delta},$$

(ii) $D_\alpha \gamma_{ik} = X_\alpha(\gamma_{ik})$ (covariant derivative with respect to ω),

(iii) $\Gamma^{\ell}{}_{ij}$ are the Christoffel symbols of the induced metric in the fibres of P. They can be computed using (2.11), (2.10) and:

(2.15)
$$X_i(\gamma_{jk}) = c_{kij} + c_{jik},$$

which is obtained by differentiating (2.2). One gets:

(2.16)
$$\Gamma^{\ell}{}_{ji} = \frac{1}{2}(c^{\ell}{}_{ij} + c_{ji}{}^{\ell} + c_{ij}{}^{\ell}).$$

Note that in the formulae (2.15) (2.16) and in the following, indices are raised or lowered by means of the metric g, that is, for latin indices, by means of γ and for greek indices, by means of $\hat{g}$.

The curvature tensor field R of the metric g can be obtained from (2.12), (2.13), (2.14) and (2.16) by using the Ricci formula:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In the following we will be interested only in the Ricci tensor Ric and the scalar curvature scal_g of g. Computing the relevant components of the curvature tensor and contracting, we obtain the components of the Ricci tensor

(2.17)
$$\begin{aligned}\mathrm{Ric}_{\alpha\beta} = \mathrm{Ric}(X_\alpha, X_\beta) =& \widehat{\mathrm{Ric}}_{\alpha\beta} - \frac{1}{2}\Omega^{\ell}{}_{\alpha\lambda}\Omega_{\ell\beta}{}^{\ell} - \frac{1}{2}c^{\ell}{}_{\ell k}\Omega^{k}{}_{\alpha\beta} - \\ & - \frac{1}{2}\gamma^{k\ell} D_\beta D_\alpha \gamma_{kl} + \frac{1}{4}\gamma_{km}\gamma^{\ell n} D_\alpha \gamma_{k\ell} D_\beta \gamma_{mn}\end{aligned}$$

$$\text{Ric}_{\alpha i} = \text{Ric}(X_\alpha, X_i) = \frac{1}{2} D^\beta \Omega_{i\alpha\beta} + \frac{1}{4}\Omega_{i\alpha\beta}\gamma^{k\ell} D^\beta \gamma_{k\ell} - \frac{1}{2} c^k{}_i{}^j D_\alpha \gamma_{jk} - \frac{1}{2} c^\ell{}_\ell{}^k D_\alpha \gamma_{ik} \tag{2.18}$$

$$\text{Ric}_{ij} = \text{Ric}(X_i, X_j) = S_{ij} + \frac{1}{4}\Omega_i{}^{\lambda\mu}\Omega_{j\lambda\mu} - \frac{1}{2} D^\alpha D_\alpha \gamma_{ij} + \frac{1}{2}\gamma^{k\ell} D^\alpha \gamma_{ik} D_\alpha \gamma_{j\ell} - \frac{1}{4}\gamma^{k\ell} D^\alpha \gamma_{k\ell} D_\alpha \gamma_{ij} \tag{2.19}$$

where:

(i) $\widehat{\text{Ric}}_{\alpha\beta}$ are the pull-back on P of the components of the Ricci tensor of $\hat{g}$,

(ii) $D_\beta D\alpha\gamma_{ij} = X_\beta(X_\alpha(\gamma_{ij})) - \hat{\Gamma}^\delta{}_{\alpha\beta} X_\delta(\gamma_{ij})$ (second order covariant derivatives).

(iii) $D_\lambda \Omega_{i\alpha\beta} = X_\lambda(\Omega_{i\alpha\beta}) - \hat{\Gamma}^\delta{}_{\alpha\lambda}\Omega_{i\delta\beta} - \hat{\Gamma}^\delta{}_{\beta\lambda}\Omega_{i\alpha\delta}$ (covariant derivatives of Ω).

(iv) S_{ij} is defined by:

$$S_{ij} = -\frac{1}{2}K_{ij} - \frac{1}{2}c^k{}_i{}^\ell c_{kj\ell} + \frac{1}{4}c_i{}^{k\ell} c_{jk\ell} - \frac{1}{2}c^\ell{}_{\ell k}(c_{ji}{}^k + c_{ij}{}^k). \tag{2.20}$$

In the last equation K_{ij} are the components of the Killing form on g .

The scalar curvature of g which was already written in [6] and [7] is equal to:

$$\text{scal}_g = \text{scal}_{\hat{g}} - \frac{1}{4}\Omega^\ell{}_{\alpha\beta}\Omega_\ell{}^{\alpha\beta} - D^\alpha(\gamma^{k\ell} D_\alpha \gamma_{k\ell}) - \frac{1}{4}\gamma^{km}\gamma^{\ell n} D^\alpha \gamma_{k\ell} D_\alpha \gamma_{mn} - \frac{1}{4}\gamma^{k\ell} D_\alpha \gamma_{k\ell}\gamma^{ij} D^\alpha \gamma_{ij} + s \tag{2.21}$$

where

$$s = S^i_i = -\frac{1}{2}K^i_i - \frac{1}{4}c^{kij}c_{kij} - c_{ji}{}^k c_{ij}{}^k.$$

The following proposition gives a property of the Ricci tensor:

PROPOSITION 1. For every *G-invariant* metric g on P :

$$\text{Ric}_{ij}\, c^i{}_k{}^j + \text{Ric}_{kj}\, c^\ell{}_\ell{}^j = D^\alpha \text{Ric}_{\alpha k} + \frac{1}{2}\gamma^{i\ell} D^\alpha \gamma_{j\ell} \text{Ric}_{\alpha k}. \tag{2.22}$$

Proof. Contracting twice the second Bianchi identity [18] one gets

$$\nabla^\alpha \mathrm{Ric}_{\alpha i} + \nabla^j \mathrm{Ric}_{ji} = \frac{1}{2} X_i(\mathrm{scal}_g). \tag{2.23}$$

Let us compute the different terms in (2.23).

(i) Since scal_g is G-invariant, the right hand side of (2.23) vanishes.

(ii) Using (2.12), (2.13) and (2.14)) we get:

$$\nabla_{X_\beta} \mathrm{Ric}_{\alpha i} = D_\beta \mathrm{Ric}_{\alpha i} + \frac{1}{2}\Omega^j{}_{\beta\alpha} \mathrm{Ric}_{ij} - \\ - \frac{1}{2}\Omega_{i\beta}{}^\lambda \mathrm{Ric}_{\alpha\lambda} - \frac{1}{2} D_\beta \gamma_{ij} \mathrm{Ric}_\alpha^j,$$

$$\nabla_{X_k} \mathrm{Ric}_{ji} = X_k(\mathrm{Ric}_{ji}) + \frac{1}{2} D^\alpha \gamma_{jk} \mathrm{Ric}_{\alpha i} + \frac{1}{2} D^\alpha \gamma_{ik} \mathrm{Ric}_{\alpha j} - \\ - \Gamma^\ell_{jk} \mathrm{Ric}_{\ell i} - \Gamma^\ell_{ik} \mathrm{Ric}_{\ell j}.$$

Substituting the above equations into (2.23) and using (2.16) we obtain:

$$0 = D^\alpha \mathrm{Ric}_{\alpha i} + \gamma^{jk} X_k(\mathrm{Ric}_{ij}) + \frac{1}{2}\gamma^{ik} D^\alpha \gamma_{jk} \mathrm{Ric}_{\alpha i} - c^k{}_k{}^j \mathrm{Ric}_{ij}.$$

From the G-invariance of the Ricci tensor we get by differentiation:

$$X_k(\mathrm{Ric}_{ij}) = \mathrm{Ric}_{j\ell} c^\ell{}_{ki} + \mathrm{Ric}_{i\ell} c^\ell{}_{kj}$$

and

$$\gamma^{jk} X_k(\mathrm{Ric}_{ji}) = \mathrm{Ric}_{j\ell} c^{\ell j}{}_i.$$

which completes the proof. ∎

3. SYMMETRY BREAKING

3.1. Field equations

We consider the geometric situation previously described in a context related to physics in the following way: the dimension of M is equal to $n = 4$ (M stands for space time), G is one of the usual internal symmetry groups of physics (in particular, it is compact). In the canonical splitting (2.4) of an invariant metric $g, \hat{g}$ is a gravitational field, ω is a gauge field (Yang-Mills field) and γ is a scalar field. The field equations for $(\hat{g}, \omega, \gamma)$ will be deduced from the action principle for the lagrangian form

$$\mathcal{L} = \mathrm{scal}_g \, v_g \tag{3.1}$$

where v_g is the volume form of the metric g on P. These equations have already been studied [4], [5] in the special case when γ is constant on P.

The case when γ is a dynamical variable with an unrestricted range was considered by Cho and Freund [6] and Scherk and Schwarz [7].

We are studying an intermediate situation by restricting the values of γ to a fixed orbit. This generalise the case when γ is constant (i.e. with values in a single point orbit). In this situation γ will be shown to disappear from the spectrum while it provides symmetry breaking and, consequently, gives mass to some of the gauge bosons defined by the Yang-Mills field ω. The field equations will reduce to equations for interacting gravitation, remaining (massless) Yang-Mills field and massive vector bosons.

Let $\mathcal{E}$ be an orbit of G in $\otimes_s^2 g^*$, consisting of positive-definite elements. To each $\beta \in \mathcal{E}$ we associate a decomposition of the Lie algebra g of G:

$$g = \mathcal{H}_\beta \oplus \mathcal{K}_\beta = \beta \tag{3.2}$$

where $\mathcal{H}_\beta$ is the Lie algebra of the stabilizer

$$H_\beta = \{a \in G | \operatorname{Ad}^*(a)\beta = \beta\} \text{ of } \beta$$

and

$$\mathcal{K}_\beta = \{\xi \in g | \beta(\xi, \eta) = 0 \ \ \forall \eta \in \mathcal{H}_\beta\} \tag{3.3}$$

is the orthocomplement of $\mathcal{H}_\beta$ in g with respect to the scalar product β.

From

$$H_{\operatorname{Ad}^*(a)\beta} = a^{-1} H_\beta \, a$$

and by orthogonality we have

$$\mathcal{K}_{\operatorname{Ad}^*(a)\beta} = \operatorname{Ad}(a^{-1})\mathcal{K}_\beta \tag{3.4}$$

We denote by $\mathcal{M}$ the set of invariant lorentzian metrics g on P such that, in their canonical splitting (2.4) the map γ takes its values in $\mathcal{E}$.

Denote by $\operatorname{Aut}(P)$ the automorphism group of the principal bundle P and let $f \in \operatorname{Aut}(P)$: f commutes with the right action of G on P, so that there exists a diffeomorphism $\hat{f}$ of M verifying $\pi \circ f = \hat{f} \circ \pi$.

Let $g \in \mathcal{M}$. Then $f^* g$ is obviously G-invariant. Moreover, if

$$g = \pi^* \hat{g} + \gamma \circ (\omega \otimes \omega)$$

is the canonincal splitting of g, we have the canonical splitting of f^*g:

$$f^*g = \pi^*\hat{f}^*\hat{g} + (f^*\gamma) \circ ((f^*\omega) \otimes (f^*\omega)).$$

We deduce that $\mathcal{M}$ is stable under the action of the group $\mathrm{Aut}(P)$.

We restrict the action principle to the metrics g belonging to $\mathcal{M}$. Then the field equation is the Einstein type equation:

$$(3.5)\qquad \begin{cases} (\mathrm{Ric}^{AB} - \frac{1}{2}\,\mathrm{scal}_g\, g^{AB})\,\delta g_{AB} = 0 \\ \\ \text{for each covariant symmetric tensor field } \delta g \text{ on } P \\ \text{which is an infinitesimal variation of } g \text{ in } \mathcal{M}. \end{cases}$$

(in (3.5) the indices A, B range from 1 to $4 + N$).

Let g be a lorentzian metric belonging to $\mathcal{M}$ with canonical splitting (2.4). Then an infinitesimal variation of g in $\mathcal{M}$ splits as follow:

$$(3.6)\qquad \delta g\ \pi^*\delta\hat{g} + \gamma \circ (\omega \otimes \delta\omega) + \gamma \circ (\delta\omega \otimes \omega) + \delta\gamma \circ (\omega \otimes \omega)$$

where $\delta\hat{g}$ is an arbitrary infinitesimal variation of $\hat{g}$, $\delta\omega$ is an arbitrary tensorial $\mathfrak{g}$-valued 1-form of type Ad on P [18] and $\delta\gamma : P \to \otimes_s^2 \mathfrak{g}^*$ is an equivariant map such that

$$(3.7)\qquad \delta\gamma(p) \in T_{\gamma(p)}\mathcal{E} \quad \forall p \in P.$$

With the notations of 2.2 we see easily that the field equations (3.5) split as follows

$$(3.8)\qquad \mathrm{Ric}_{\alpha\beta} - \frac{1}{2}\,\mathrm{scal}_g\, g_{\alpha\beta} = 0$$

$$(3.9)\qquad \mathrm{Ric}_{i\alpha} = 0$$

$$(3.10)\qquad \begin{cases} (\mathrm{Ric}^{ij} - \frac{1}{2}\,\mathrm{scal}_g\, \gamma^{ij})\,\delta\gamma_{ij} = 0 \\ \\ \text{for each equivariant map } \delta\gamma : P \to \otimes_s^2 \mathfrak{g}^* \text{ satisfying (3.7).} \end{cases}$$

PROPOSITION 2. *The equation* (3.10) *is a consequence of the equation* (3.9).

Proof. Let $\delta\gamma : P \to \otimes_s^2 \mathfrak{g}^*$ be an equivariant map such that $\delta\gamma(p) \in T_{\gamma(p)}\mathcal{E}$.

From (3.2) we deduce that for each $p \in P$ there exists a unique $\lambda(p) \in \mathcal{K}_{\gamma(p)}$ such that

$$\delta\gamma(p) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}^*(\exp(t\lambda(p)))\gamma(p)\,|_{t=0}$$

Using the equivariance of $\delta\gamma$ and (3.4) it is easy to show that the map $\lambda : P \to g$ is also equivariant.

Introducing a basis $(e_1, \dots, e_N)$ of g we have $\lambda(p) = \lambda^i(p) e_i$ and

$$\delta\gamma_{ij}(p) = \delta\gamma(p)(e_i, e_j) =$$
$$= \gamma(p)([\lambda(p), e_i], e_j) + \gamma(p)(e_i, [\lambda(p), e_j]).$$

Therefore

$$\delta\gamma_{ij}(p) = \lambda^k(p)(c_{ikj} + c_{jik}). \tag{3.11}$$

Substituting (3.11) into the left hand side of (3.10) we obtain:

$$(\mathrm{Ric}^{ij} - \frac{1}{2}\,\mathrm{scal}_g\,\gamma^{ij})\delta\gamma_{ij} = 2\,\lambda^k(\mathrm{Ric}_{ij}\,c^i{}_k{}^j + \frac{1}{2}\,\mathrm{scal}_g\,c^\ell{}_{\ell k}). \tag{3.12}$$

The conclusion of the proof follows easily from (2.22), (3.12) and the compactness of G (what implies that all the $c^\ell{}_{\ell k} = \mathrm{Tr}(\mathrm{ad}(e_k))$ vanish). ■

3.2. Decomposition of the Yang-Mills field

In the previous section, we have found that for invariant metrics $g \in \mathcal{M}$, Einstein equations reduce to (3.8) and (3.9). In the following, we will show that the derivatives of the scalar field γ disappear from these equations. Meanwhile, part of the gauge field appear to be massive.

Let $g \in \mathcal{M}$ with canonical splitting $g = \pi^*\hat{g} + \gamma \circ (\omega \otimes \omega)$. The scalar part γ of g defines a foliation of P whose leaves $Q_\beta = \gamma^{-1}(\beta), \beta \in \mathcal{E}$, are principal subbundles of $P \to M$. The structure group of the principal bundle Q_β is the stabilizer H_β of β. We associate to this foliation a splitting of the gauge field ω as follows:

PROPOSITION 3.

1. *There exists a unique decomposition*

$$\omega = \omega' + \sigma \tag{3.13}$$

where:

(i) ω' *is a connection form on* P *such that its restriction to the tangent space at a point* p *to the leaf* $Q_{\gamma(p)}$ *containing* p *takes its values in* $\mathcal{H}_{\gamma(p)}$.

(ii) $\sigma \in \Omega^1(P, g)$ *is as tensorial* 1*-form such that, for each* $p \in P$, $\sigma(p)$ *takes its values in* $\mathcal{K}_{\gamma(p)}$.

2. *The horizontal space at any point* $p \in P$ *for the connection* ω' *is contained in the tangent space to the leaf* $Q_{\gamma(p)}$.

3. *The pull-back of the connection* ω' *to any of the leaves* Q_β *is a connection form on the bundle* Q_β.

Proof. We make use of the decomposition

$$T_pP = T_pQ_{\gamma(p)} \oplus V_p \tag{3.14}$$

where V_p is the subspace of the vertical space at p defined by

$$V_p = \left\{ \frac{\mathrm{d}}{\mathrm{d}t} p\exp(t\xi)|_{t=0} \mid \xi \in \mathcal{K}_{\gamma(p)} \right\}. \tag{3.15}$$

The uniqueness of the decomposition (3.13) is directly derived from (3.14).

Further we give below an explicit expression for ω': for $\beta \in \mathcal{E}$, let $u_\beta : g \to \mathcal{H}_\beta$ be the projection with kernel $\mathcal{K}_\beta$ and, for $p \in P$, let $v_p : T_pP \to V_p$ be the projection with kernel $T_pQ_{\gamma(p)}$. The 1-form $\omega' \in \Omega^1(P, g)$ given by

$$\omega'_p = u_{\gamma(p)} \circ \omega_p + \omega_p \circ v_p \tag{3.16}$$

is easily shown to satisfy the properties of connections [17]; for this purpose one can use the following formulae:

$$(\omega_p \circ v_p)\left(\frac{\mathrm{d}}{\mathrm{d}t} p\exp(t\xi)|_{t=0}\right) = \xi - u_{\gamma(p)}(\xi) \qquad p \in P, \quad \xi \in g. \tag{3.17}$$

$$u_{\mathrm{Ad}^*(a)\beta} = \mathrm{Ad}(a^{-1}) \circ u_\beta \circ \mathrm{Ad}(a) \qquad \beta \in \mathcal{E}, \quad a \in G. \tag{3.18}$$

$$v_{pa} \circ T_pR_a = T_pR_a \circ v_p \qquad p \in P, \quad a \in G. \tag{3.19}$$

(we recall that $R_a : p \in P \mapsto pa \in P$ is the right translation on P by $a \in G$).

It is obvious from (3.16) that ω' and $\sigma = \omega - \omega'$ verify the conditions of the proposition.

Finally, 2 is easily deduced from (3.14) and (3.16) and 3 is a consequence of the definition of ω'. ∎

With notations of proposition 3, the connection ω' will be called the *residual Yang-Mills field* and the tensorial form σ will describe *vector gauge bosons* which will be shown further to be massive.

Let $(X_\alpha, X_i)_{\substack{1\le\alpha\le 4\\ 5\le i\le N+4}}$ be the local frame field considered in section 2.2. Using the proposition 3, we compute the covariant derivatives $D_\alpha\gamma_{ij}$ as follows: first we notice that the horizontal (for ω) vector $X_\alpha(p)$ admits the decomposition:

$$X_\alpha(p) = X'_\alpha(p) + \frac{\mathrm{d}}{\mathrm{d}t} p\exp(-t\sigma(X_\alpha)(p))|_{t=0} \tag{3.20}$$

where X'_α is the horizontal lift according to the connection ω' of the vector field $\dfrac{\partial}{\partial x^\alpha}$ (see [12]). Since $X'_\alpha(\gamma) = 0$ (proposition 3, part 2) we deduce from (3.20):

$$\begin{aligned} D_\alpha\gamma_{ij}(p) &= X_\alpha(\gamma)(p)(e_i, e_j) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(p\exp(-t\sigma(X_\alpha)(p)))|_{t=0}(e_i, e_j) = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma(p)(\mathrm{Ad}(\exp(-t\sigma(X_\alpha)(p)))e_i, \mathrm{Ad}(\exp(-t\sigma(X_\alpha)(p)))e_j)|_{t=0} = \\ &= -\gamma(p)([\sigma(X_\alpha)(p), e_i], e_j) - \gamma(p)(e_i, [\sigma(X_\alpha)(p), e_j]). \end{aligned}$$

Let us write:

$$\sigma(X_\alpha) = \sigma^k{}_\alpha e_k, \tag{3.21}$$

then we have:

$$D_\alpha \gamma_{ij} = -\sigma^k{}_\alpha (c^\ell{}_{ki}\gamma_{\ell j} + c^\ell{}_{kj}\gamma_{i\ell}). \tag{3.22}$$

Contracting (3.22) with γ^{-1} we obtain:

$$\gamma^{ij} D_\alpha \gamma_{ij} = -2\,\mathrm{Tr}(\mathrm{ad}(\sigma(X_\alpha))) = 0. \tag{3.23}$$

Equation (3.22) implies that the kinetic energy of the Higgs field γ can be expressed in terms of the tensorial form σ and thus disappear from the lagrangian scal_g. Moreover, the formula (3.23) allows simplifications of the field equations (3.8), (3.9) for the dynamical variables $(\hat{g}, \omega)$ which can be written, using (2.17), (2.18) and (2.21):

$$\widehat{\mathrm{Ric}}_{\alpha\beta} - \frac{1}{2}\Omega^\ell{}_{\alpha\lambda}\Omega_{\ell\beta}{}^\lambda - \frac{1}{4}\gamma^{km}\gamma^{\ell n} D_\alpha\gamma_{k\ell} D_\beta\gamma_{mn} - \frac{1}{2}\,\mathrm{scal}_g\,\hat{g}_{\alpha\beta} = 0 \tag{3.24}$$

$$D^\beta \Omega_{i\alpha\beta} - c^k{}_i{}^j D_\alpha \gamma_{jk} = 0 \tag{3.25}$$

where

$$\begin{aligned} \mathrm{scal}_g = \mathrm{scal}_{\hat{g}} &- \frac{1}{4}\Omega^\ell{}_{\lambda\mu}\Omega_\ell{}^{\lambda\mu} - \frac{1}{4}\gamma^{km}\gamma^{\ell n} D^\lambda\gamma_{k\ell} D_\lambda\gamma_{mn} - \\ &- \frac{1}{2}K^i{}_i - \frac{1}{4}c^{kij}c_{kij} \end{aligned} \tag{3.26}$$

According to proposition 3, we will consider, from now on, $(\hat{g}, \omega', \sigma)$ as the dynamical variables corresponding to the Kaluza-Klein field g submitted to the constraint which forces γ to take its values in the G-orbit $\mathcal{E}$. In order to write the field equation, for these dynamical variables, we have to express the curvature of ω and its covariant derivatives in terms of ω' and σ. We have:

$$\begin{aligned} \Omega_{\alpha\beta} &= \Omega(X_\alpha, X_\beta) = \Omega(X'_\alpha, X'_\beta) = \\ &= d\,\omega(X'_\alpha, X'_\beta) + [\omega(X'_\alpha), \omega(X'_\beta)] = \\ &= d\,\omega'(X'_\alpha, X'_\beta) + d\,\sigma(X'_\alpha, X'_\beta) + [\sigma(X'_\alpha), \sigma(X'_\beta)]. \end{aligned}$$

Hence

$$\Omega_{\alpha\beta} = \Omega'_{\alpha\beta} + (d^{\omega'}\sigma)_{\alpha\beta} + [\sigma(X_\alpha), \sigma(X_\beta)] \tag{3.27}$$

or

$$\Omega^j{}_{\alpha\beta} = \Omega'^j{}_{\alpha\beta} + \mathrm{d}^{\omega'}\sigma^j{}_{\alpha\beta} + c^j{}_{k\ell}\sigma^k{}_\alpha\sigma^\ell{}_\beta \tag{3.28}$$

where $\mathrm{d}^{\omega'}\sigma$ is the exterior covariant differential of σ with respect to ω'.

On the other hand, let D' denote the covariant derivation with respect to the connection ω'. Then, using (3.20) we have (for any $p \in P$):

$$D_\lambda\Omega_{\alpha\beta}(p) - D'_\lambda\Omega_{\alpha\beta}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Omega_{\alpha\beta}(p\exp(-t\sigma(X_\lambda)(p)))|_{t=0}. \tag{3.29}$$

Since the vectors X_α are invariant by the right action of G on P, the curvature components $\Omega_{\alpha\beta}$ satisfy:

$$\Omega_{\alpha\beta}(pa) = \mathrm{Ad}\,(a^{-1})\Omega_{\alpha\beta}(p) \qquad p \in P, \quad a \in G. \tag{3.30}$$

Reporting (3.30) into (3.29) one gets:

$$D_\lambda\Omega_{\alpha\beta} = D'_\lambda\Omega_{\alpha\beta} + [\sigma(X_\lambda), \Omega_{\alpha\beta}]. \tag{3.31}$$

We deduce from (3.31) and (3.22) that:

$$D^\beta\Omega_{i\alpha\beta} = \gamma_{ij}(D'^\beta\Omega^j{}_{\alpha\beta} - \sigma^k{}_\beta c^j{}_{\ell k}\Omega^\ell{}_\alpha{}^\beta). \tag{3.32}$$

Hence, the field equation (3.25) can be written:

$$D'^\beta\Omega^j{}_{\alpha\beta} - \sigma^k{}_\beta c^j{}_{\ell k}\Omega^\ell{}_\alpha{}^\beta - c^{kj\ell}D_\alpha\gamma_{k\ell} = 0. \tag{3.33}$$

Combined with (3.28) and (3.22), (3.33) gives the field equations for the dynamical variables ω' and σ.

To make it clearer we restrict to a subbundle $Q_\beta (\beta \in \mathcal{E})$ and we choose the basis $(e_1, \dots, e_N)$ of the Lie algebra g in such a way, that it is adapted to the decomposition (3.2). We denote $(e_{i_1})_{5\le i_1\le 4+N-d}$ the basis vectors which belong to $\mathcal{H}_\beta$ and $(e_{i_2})_{5+N-d\le i_2\le 4+N}$ those belonging to $\mathcal{K}_\beta$ ($d = \dim \mathcal{E}$). The following formulae are obvious consequences of this choice of basis:

$$\begin{cases} \beta_{i_1 j_2} = c^{i_2}{}_{j_1 k_1} = c^{i1}{}_{j_1 k_2} = 0 \\ c_{kj_1\ell} + c_{\ell j_1 k} = 0 \end{cases} \tag{3.34}$$

(recall that indices k, ℓ, without subscripts range from 5 to $4 + N$).

Making use of proposition 2 and of formulae (3.22), (3.28) and (3.34), we obtain, on Q_β, a splitting of equation (3.33) into equations (3.35) and (3.37) written below:

$$D'^\beta \Omega'^{i_1}{}_{\alpha\beta} = \mathbf{j}^{i_1}{}_\alpha \tag{3.35}$$

where $(\mathbf{j}^{i_1}_\alpha)$ are the components of a current 1-form (with values in $\mathcal{H}_\beta$) associated to the matter field defined by σ as follows:

$$\begin{aligned} \mathbf{j}^{i_1}{}_\alpha = & -\sigma^{k_2}{}_\alpha (c^{j_1}{}_{k_2\ell_2} D'^\beta \sigma^{\ell_2}{}_\beta - c^{\ell_2}{}_{k_2 n_2} c_{\ell_2 m_2}{}^{j_1} \sigma^{m_2}{}_\beta \sigma^{n_2 \beta}) + \\ & + \sigma^{k_2}{}_\beta (c_{\ell_2 k_2}{}^{j_1} \mathrm{d}^{\omega'} \sigma^{\ell_2}{}_\alpha{}^\beta + c^{j_1}{}_{k_2\ell_2} D'^\beta \sigma^{\ell_2}{}_\alpha). \end{aligned} \tag{3.36}$$

$$\begin{aligned} & D'^\beta \mathrm{d}^{\omega'} \sigma^{j_2}{}_{\alpha\beta} + c^{k j_2 \ell}(c_{k i_2 \ell} + c^{\ell i_2 k}) \sigma^{i_2}{}_\alpha = \\ & = c^{j_2}{}_{k_2\ell_2} \sigma^{\ell_2}{}_\alpha D'^\beta \sigma^{k_2}{}_\beta + \sigma^{k_2}{}_\beta (c_{\ell_1 k_2}{}^{j_2} \Omega'^{\ell_1}{}_\alpha{}^\beta + c_{\ell_2 k_2}{}^{j_2} \mathrm{d}^{\omega'} \sigma^{\ell_2}{}_\alpha{}^\beta - \\ & - c^{j_2}{}_{k_2\ell_2} D'^\beta \sigma^{\ell_2}{}_\alpha + c_{\ell k_2}{}^{j_2} c^\ell{}_{m_2 n_2} \sigma^{m_2}{}_\alpha \sigma^{n_2 \beta}). \end{aligned} \tag{3.37}$$

One recognizes in (3.35) the standard gauge field equation for the residual Yang-Mills field ω' and in (3.37) the field equation for the vector gauge bosons defined by the tensorial form σ. We notice that all terms in the right hand side of (3.37) are non linear in (ω', σ) and correspond to interactions between ω' and σ, or to self interactions of σ. Hence, all the informations about the mass of the vectorial bosonic field σ are contained in the second term of the left hand side of (3.37). This will be discussed in the next subsection.

3.3. Masses of vector gauge bosons

For any $\beta \in \mathcal{E}$, and for any linear endomorphism $f \in \mathcal{L}(g)$, let $f'_\beta \in \mathcal{L}(g)$ denote the adjoint endomorphism of f with respect to the scalar product β in g

$$\beta(f\xi, \eta) = \beta(\xi, f'_\beta \eta) \qquad \xi, \eta \in g.$$

We associate with each $\beta \in \mathcal{E}$ a bilinear form $B(\beta)$ on g defined by:

$$B(\beta)(\xi, \eta) = K(\xi, \eta) + \mathrm{Tr}(\mathrm{ad}(\eta)'_\beta \circ \mathrm{ad}(\xi)), \tag{3.38}$$

where K is the Killing form of g.

Using the coefficients $\beta_{k\ell} = \beta(e_k, e_\ell)$ of the matrix of β associated to the basis $(e_1, \dots, e_N)$ of g and the coefficients $\beta^{k\ell}$ of the inverse matrix, we have:

$$B(\beta)(\xi, \eta) = K(\xi, \eta) + \beta^{k\ell} \beta([\xi, e_k], [\eta, e_\ell]) \tag{3.39}$$

and

$$B(\beta)_{ij} = B(\beta) e_i, e_j) = K_{ij} + \beta^{k\ell} \beta_{mn} c^m{}_{ik} c^m{}_{j\ell}. \tag{3.40}$$

PROPOSITION 4.

1. *For each* $\beta \in \mathcal{E}$ *, the bilinear form* $B(\beta)$ *is symmetric non negative and its kernel is the Lie subalgebra* $\mathcal{H}_\beta$ *.*

2. *For* $\beta \in \mathcal{E}$ *and* $\beta' = \mathrm{Ad}^*(a)(\beta) \in \mathcal{E}$ *the bilinear forms* $B(\beta)$ *and* $B(\beta')$ *are related by:*

$$B(\beta') = \mathrm{Ad}^*(a)\, B(\beta).$$

Proof.

1. The symmetry of $B(\beta)$ is obvious from (3.39).

Since $K(\xi,\xi) = \mathrm{Tr}((\mathrm{ad}(\xi))^2) = \mathrm{Tr}((\mathrm{ad}(\xi)'_\beta)^2)$ we can write

$$B(\beta)(\xi,\xi) = \frac{1}{2}\mathrm{Tr}((\mathrm{ad}(\xi) + \mathrm{ad}(\xi)'_\beta)^2). \tag{3.41}$$

It follows from (3.41) that $B(\beta)(\xi,\xi) \geq 0$ and that

$$\mathrm{Ker}(B(\beta)) = \{\xi \in g \,|\, \mathrm{ad}(\xi) + \mathrm{ad}(\xi)'_\beta = 0\} = \mathcal{H}_\beta.$$

2. One obtains easily, for $f \in \mathcal{L}(g), \beta \in \mathcal{E}, a \in G$:

$$f'_{\mathrm{Ad}^*(a)\beta} = \mathrm{Ad}(a^{-1}) \circ (\mathrm{Ad}(a) \circ f \circ \mathrm{Ad}(a^{-1}))'_\beta \circ \mathrm{Ad}(a).$$

When $f = \mathrm{ad}(\xi), \xi \in g$, this formula can be written:

$$\mathrm{ad}(\xi)'_{\mathrm{Ad}^*(a)\beta} = \mathrm{Ad}(a^{-1}) \circ \mathrm{ad}(\mathrm{Ad}(a)\xi)'_\beta \circ \mathrm{Ad}(a). \tag{3.42}$$

Using the G-invariance of the Killing form K and (3.42) we get:

$$\begin{aligned} B(\mathrm{Ad}^*(a)\beta)(\xi,\eta) &= K(\mathrm{Ad}(a)\xi, \mathrm{Ad}(a)\eta) + \\ &\quad + \mathrm{Tr}(\mathrm{ad}(\mathrm{Ad}(a)\eta)'_\beta \circ \mathrm{ad}(\mathrm{Ad}(a)\xi)) = \\ &= B(\beta)(\mathrm{Ad}(a)\xi, \mathrm{Ad}(a)\eta) \qquad \text{Q.E.D.} \end{aligned}$$

■

Let $\overset{(2)}{M}(\beta) \in \mathcal{L}(g)$ be the linear endomorphism of g associated to $\beta \in \mathcal{E}$ by:

$$B(\beta)(\xi,\eta) = \beta(\xi, \overset{(2)}{M}(\beta)\eta).$$

From proposition 4 (part 1), we deduce that $\overset{(2)}{M}(\beta)$ stabilizes $\mathcal{K}_\beta$ and that the eigenvalues of the restriction of $\overset{(2)}{M}(\beta)$ to $\mathcal{K}_\beta$ are positive. On the other hand, it follows from

(3.40) that the coefficient of $\sigma^{i_2}{}_\alpha$ in the second term of the left hand side of (3.37) is the general component of the matrix which represents the restriction of $\overset{(2)}{M}(\beta)$ to $\mathcal{K}_\beta$ in the basis $(e_{j_2})_{5+N-d\leq j_2\leq 4+N}$. Choosing these basis vectors to be eigen vectors for $\overset{(2)}{M}(\beta)$:

$$\overset{(2)}{M}(\beta)e_{j_2} = (m_{j_2})^2 e_{j_2}, \qquad m_{j_2} > 0,$$

the left hand side of (3.37) takes the following simple form:

$$D'^{\beta}\mathrm{d}^{\omega'}\sigma^{j_2}{}_{\alpha\beta} + (m_{j_2})^2\sigma^{j_2}{}_{\alpha}$$

Hence, the mass distribution of the vector gauge bosons defined by σ is given by the square roots of the non-vanishing eigenvalues of $\overset{(2)}{M}(\beta)$. It follows from the proposition 4 (part 2) that this mass distribution does not depend on the point β of the orbit $\mathcal{E}$ which is used for computations and consequently, that it is intrinsically given by the orbit $\mathcal{E}$.

4. AN EXAMPLE

In this section we consider the theory introduced before in the case when the internal symmetry group is $\mathrm{Spin}(4)$ and where subbundles associated to the scalar part γ of a Kaluza-Klein metric g (section 3.2) have structure groups isomorphic to $U(1)$. The group $\mathrm{Spin}(4)$ is known to be the product of two simple normal subgroups isomorphic to $SU(2)$ [20].

The mass distribution of the matter field σ corresponding to a solution of field equations (3.24), (3.35), (3.37), is determined by the choice of a 5-dimensional orbit $\mathcal{E}$ of $\mathrm{Spin}(4)$ in $\otimes^2_s g^*$. Such an orbit is obtained in the following way.

Let H be a connected, 1-dimensional, closed subgroup of $\mathrm{Spin}(4)$, not contained in any of the $SU(2)$ factors (this assumption is made in order to obtain as a many charged vector gauge bosons as possible).

For each scalar product $\beta \in \otimes^2_s(\mathrm{spin}(4))^*$ such that $H_\beta = H$, we can take $\mathcal{E} = \mathrm{Ad}^*(\mathrm{Spin}(4))\beta$. In order to determine the scalar products satisfying $H_\beta = H$ we choose a basis $(e_1,\ldots,e_6)$ of $\mathrm{spin}(4)$ verifying:

(i)
$$[e_1,e_2] = e_3,\ [e_2,e_3] = e_1,\ [e_3,e_1] = e_2$$
$$[e_4,e_5] = e_6,\ [e_5,e_6] = e_4,\ [e_6,e_4] = e_5$$
$$[e_i,e_j] = 0 \qquad \text{for } 1\leq i\leq 3,\ 4\leq j\leq 6$$

(ii) there exists a real number $\alpha > 0$ such that:

$$Q = e_3 + \alpha e_6$$

is a generator of the Lie algebra $\mathcal{H}$ of H.

A straightforward computation shows that a scalar product β on spin(4) is $\mathrm{Ad}^*(H)$-invariant if and only if its coefficients $\beta_{ij} = \beta(e_i, e_j)$ verify:

$$\beta_{12} = \beta_{13} = \beta_{23} = \beta_{34} = \beta_{35} = \beta_{16} = \beta_{26} = \beta_{45} = \beta_{46} = \beta_{56} = 0 \tag{4.1}$$

$$\beta_{11} - \beta_{22} = \beta_{44} - \beta_{55} = 0 \tag{4.2}$$

$$\beta_{24} + \alpha\beta_{15} = \beta_{15} + \alpha\beta_{24} = \beta_{14} - \alpha\beta_{25} = \beta_{25} - \alpha\beta_{14} = 0. \tag{4.3}$$

We deduce from conditions (4.3) that $\alpha = 1$ is necessary for $H_\beta = H$ (if $\alpha \neq 1$, these conditions are equivalent to $\beta_{14} = \beta_{15} = \beta_{24} = \beta_{25} = 0$, and H_β is easily seen to contain the 2-dimensional, abelian subgroup of Spin(4) generated by (e_3, e_6)).

Assuming $\alpha = 1$ we obtain $H_\beta = H$ if:

$$0 < (\beta_{14})^2 + (\beta_{15})^2 \neq (\beta_{36})^2. \tag{4.4}$$

Therefore there is a family of scalar products β satisfying $H_\beta = H$, depending on the 7 parameters $(\beta_{11}, \beta_{33}, \beta_{44}, \beta_{66}, \beta_{14}, \beta_{15}, \beta_{36})$ satisfying (4.4). Notice that, since the centralizer of H in Spin(4) is 2-dimensional, we can assume, without loss of generality on the orbit $\mathcal{E} = \mathrm{Ad}^*(\mathrm{Spin}(4))\beta$ that, moreover, $\beta_{15} = 0$.

From now on, we choose a positive scalar product β on spin(4) such that its components in the basis $(e_1, \ldots, e_6)$ verify:

$$\begin{cases} \beta_{12} = \beta_{13} = \beta_{15} = \beta_{16} = \beta_{23} = \beta_{24} = 0, \\ \beta_{26} = \beta_{34} = \beta_{35} = \beta_{45} = \beta_{46} = \beta_{56} = 0 \\ \beta_{11} = \beta_{22},\ \beta_{14} = \beta_{25},\ \beta_{44} = \beta_{55}, \end{cases} \tag{4.5}$$

$$0 \neq |\beta_{14}| \neq |\beta_{36}|. \tag{4.6}$$

A straightforward computation, using (3.40), show that the bilinear form $B(\beta)$ defined on spin(4) by (3.38) satisfies also the conditions (4.5).

Therefore, the mass endomorphism $\overset{(2)}{M}(\beta)$ defined in (3.43) stabilizes the three Cartan subalgebras C_1, C_2, C_3, of spin(4), respectively spanned by $\{e_1, e_4\}, \{e_2, e_5\}, \{e_3, e_6\}$. Moreover, the restrictions of $\overset{(2)}{M}(\beta)$ to C_1 and C_2 admit the same matrix in the basis (e_1, e_4) and (e_2, e_5) respectively. It follows that $C_1 \oplus C_2$ is the direct sum of two 2-dimensional eigenspace of $\overset{(2)}{M}(\beta)$ with eigenvalues $(m_1)^2$ and $(m_2)^2$. On the other hand, the intersection $\mathcal{K}_\beta \cap C_3$ is also an eigenspace of $\overset{(2)}{M}(\beta)$ with eigenvalue

$(m_3)^2$. Since non-vanishing elements of $C_1 \oplus C_2$ do not commute with $Q = e_3 + e_6$, we can conclude that the choice of the orbit $\mathcal{E}$ for the constraint on the vertical part of Kaluza-Klein metric g generates a matter field σ which is composed of:

(i) a pair of charged vector bosons, each mass being equal to m_1
(ii) a pair of charged vector bosons, each mass being equal to m_2
(iii) a single neutral vector boson with mass equal to m_3 .

5. CONCLUDING REMARKS

5.1. The mechanism discussed in section 3 to get symmetry breaking from Kaluza-Klein metrics cannot be applied to the standard model, where the symmetry group is $G = SU(2) \times U(1)$ and where the symmetry is broken to a connected, 1-dimensional subgroup different from the center of G .

This is due to the fact that, in this case, the stabilizer H_β of any scalar product β on g have to contain the center $U(1)$ of G . However it seems possible that this procedure could be used to study the «leptoquark bosons» in the $SU(5)$ Model [13].

5.2. Let us go back to section 3 where an orbit $\mathcal{E}$ of G in $\otimes_s^2 g^*$ has been selected. Then it is easy to associate with any metric $g \in \mathcal{M}$ on the principal bundle P , a Witten *ansatz* [11] for Kaluza-Klein theory on an associated bundle with fiber $\mathcal{E}$.

On the other hand, one can notice that the mechanism of mass creation described above relies on the following fact: the vertical part of a Kaluza-Klein metric on a principal bundle can be considered as a Higgs field. It is possible to generalize the Witten *ansatz* in such a way that the metric on an associated bundle with fiber G/H (reductive homogeneous space) contains a map $\gamma : P \to \otimes_s^2 g^*$. But, in general, and since the subgroup H is not normal in G , this map γ cannot be G-equivariant. Therefore, it does not seem possible to generalize our procedure to the Kaluza-Klein theory on an associate bundle with fiber a reductive homogeneous space.

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